INDUCED AUTOMORPHISMS OF FREE GROUPS AND FREE METABELIAN GROUPS

BY
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1. **Introduction.** This paper studies the automorphisms of a group G, which is either a free group F or the free metabelian group $\Phi = F/F''$. In particular we are concerned with which automorphisms of the group G/G_n, where G_n is the nth term of the lower central series for G, are induced by automorphisms of G. The following is a description of the main results. We assume in all cases that G has finite rank q $(q \geqq 2)$ and throughout this introduction G will refer to one of the above groups.

We begin with the following Theorem which is also proved in Andreadakis [1].

THEOREM 1. *Any automorphism of G/G_3 is induced by an automorphism of G.*

With respect to the lower central series, this is the best positive result because for $n \geqq 4$, there exists an automorphism of G/G_n which is not induced by an automorphism of G. This is a corollary of the following main theorem of this paper.

THEOREM 2. *Let Φ have finite rank q. Then for $n \geqq 4$, the group of automorphisms of Φ/Φ_n which induces the identity automorphism in Φ/Φ_{n-1} and which is induced by automorphisms of Φ is a free abelian group of rank*

$$q(n-2)\binom{n+q-3}{q-2} - \binom{n+q-3}{q-1}.$$

One can compare this number with the rank of the group of all automorphisms of Φ/Φ_n which induces the identity automorphism in Φ/Φ_{n-1}. For example, if $q=3$, then the former rank is $\frac{1}{2}n(5n-11)$ while the latter is $3n(n-2)$. These computations are done at the end of §5, where a table is given comparing small values of n. However, the interest in Theorem 2 is not so much the actual knowledge of the rank, but the fact that the method of proof gives detailed information as to which automorphisms of Φ/Φ_n of the above form cannot be induced by automorphisms of Φ. For example, our knowledge is so precise that we are able to prove the following: We assume that the naturally isomorphic groups F/F_4 and Φ/Φ_4 are identified, and thus there exists natural homomorphisms $F \to \Phi \to \Phi/\Phi_4$.

THEOREM 4. *Every automorphism of Φ/Φ_4 which is induced by an automorphism of Φ is induced by an automorphism of F.*

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In realizing the significance of Theorem 4, one should bear in mind that Theorem 2 tells us that a large number of automorphisms of Φ/Φ_4 are not induced by automorphisms of Φ. Also it is not known at present whether every automorphism of Φ is induced by an automorphism of F.

Theorem 1 is proved by invoking a very important theorem due to J. Nielsen and W. Magnus, which we now proceed to describe. Let us call an automorphism of G which induces the identity automorphism in the abelianized group an IA-automorphism of G. The fact that every automorphism of G/G_2 is induced by an automorphism of G makes it possible, at least in questions concerned with induced automorphisms, to confine attention to the IA-automorphisms. Now in the case where the rank of G is two, the situation is very pleasant owing to the fact that the only IA-automorphisms of G are inner automorphisms (J. Nielsen [11] and S. Bachmuth [2]). In the case where the rank of G is larger than two, the situation is much more complex. Here we have the following theorem due to J. Nielsen [12] and W. Magnus [6]: Suppose F is freely generated by $a_1, a_2, \cdots, a_q$ $(q \geqq 3)$. Then the IA-automorphism group of F is generated by the following automorphisms:

$$k_{ijl}: \begin{array}{ll} a_i \to a_i a_j a_l a_j^{-1} a_l^{-1}, & i \neq j \neq l \neq i, \\ a_r \to a_r, & r \neq i, \end{array}$$

and

$$k_{ij}: \begin{array}{ll} a_i \to a_j a_i a_j^{-1}, & i \neq j, \\ a_r \to a_r, & r \neq i. \end{array}$$

(A corresponding result for the IA-automorphism group of Φ is at present an open problem and appears to be very difficult.)

The proof of Theorem 2 starts with a matrix representation of the IA-automorphism group of Φ due to W. Magnus. The computational possibilities inherent in the representation are enough to derive Theorem 2 despite the fact that there is no known analogue to the Nielsen-Magnus theorem for Φ. On the other hand there are no results comparable to Theorem 2 for the free group when $n > 4$. For $n = 4$, as in the case of $n = 3$, the Nielsen-Magnus theorem plays a prominent role. (See however [1], and the remarks concerning it at the end of this introduction.)

Theorem 2 is derived by representing the group of induced automorphisms by an additive group of matrices and then effecting a further reduction of this latter group to a system of linear equations over the integers. In the course of the proof of Theorem 2, there appears a minor contribution to the problem of determining a set of generators for the IA-automorphism group of Φ. Specifically, let A denote the group of automorphisms of Φ/Φ_n which induces the identity in Φ/Φ_2 and which is induced by automorphisms of Φ. Then various sets of automorphisms of Φ are constructed which can serve as generating sets for those automorphisms inducing the group A. At the moment we state this as follows. A more explicit formulation of Theorem 3, where a particular set of β's are described, is given at the end of §5.

THEOREM 3. *Let α be an IA-automorphism of Φ and let n be any positive integer. Then there exists a finite number of IA-automorphisms $\beta_1, \cdots, \beta_j$ each of which leaves all but (at most) two generators of Φ fixed and such that the automorphism $\beta = \beta_j \cdots \beta_2 \beta_1 \alpha$ satisfies $g\beta \equiv g \pmod{\Phi_n}$ for all g in Φ.*

Since this manuscript was submitted, there has appeared the paper of S. Andreadakis [1]. There is some overlap with this paper as for example, Andreadakis also proves our Theorem 1 and Lemma 5 of §4 (cf. Theorem 6.2 in [1]), and exhibits an automorphism of F/F_4 which is not induced from an automorphism of F. However [1] is an excellent complement to this paper since in it Andreadakis undertakes a program for the free group F which closely parallels the program for $\Phi = F/F''$ undertaken here. And even in the situations where results overlap, the methods are usually different.

Theorem 1 is proved in §4. Theorems 2,3 and 4 occupy §§5 and 6 respectively. The results in this paper are from my thesis written at New York University under the guidance of Professor W. Magnus. I would like to express once again my gratitude to Professor Magnus for his constant help and guidance. I would also like to take this opportunity to thank Professor G. Baumslag for many helpful discussions and to thank Professor R. Lyndon for his many helpful comments which have led to numerous simplifications of my original exposition.

2. **Notation and preliminaries.** Let G be a group. x^y will mean yxy^{-1} and $[x,y]$ will mean $xyx^{-1}y^{-1}$ for any elements x, y of G, and inductively an m-fold commutator is defined by

$$[x_1, \cdots, x_m, x_{m+1}] = [[x_1, \cdots, x_m], x_{m+1}].$$

If A, B are subgroups of G, $[A,B]$ is the subgroup of G generated by all commutators of the form $[a,b]$ where $a \in A$, $b \in B$. The commutator subgroup of G is $G' = [G,G]$, and the second commutator subgroup of G is $G'' = [G',G']$. The nth term of the lower central series of G is G_n, where $G_1 = G$ and for $n > 1$, $G_n = [G_{n-1}, G]$.

Throughout this paper the letter F will be reserved to denote a free group and Φ will denote the free metabelian group $\Phi = F/F''$.

Let F be freely generated by $a_1, a_2, \cdots, a_q$. Let F^* denote the corresponding free Magnus ring; i.e., the associative ring of power series over the integers in noncommuting indeterminates $x_1, \cdots, x_q$. F can be embedded isomorphically in F^* by the correspondence $a_i \to 1 + x_i$, $a_i^{-1} \to \sum_0^\infty (-1)^m x_i^m$ (Magnus [7]). A monomial in the x_i of degree m is called a term of dimension m. The dimension of an element $g \neq 1$ in F is the degree of the smallest (nonconstant) monomial occurring in the representation of g. The following holds: If $1 \neq g$ is in F, then g is in F_n if and only if dim $g \geqq n$ (Magnus [7]). We will refer to this as Magnus' Theorem.

Let F have rank q. Then F_n/F_{n+1} is a free abelian group of rank $\Psi_q(n) = 1/n \sum q^d \mu(n/d)$, where the summation runs over all divisors d of n and μ is the Moebius function (Witt [13]).

Let Φ be of rank q with free generators $a_1, \cdots, a_q$. A set of generators for the commutator subgroup Φ' of Φ is given by all commutators of the form

$$[a_i, a_j]^{a_1^{j_1} \cdots a_q^{j_q}},$$

where $i < j$ and the j_i's are arbitrary integers. We denote the integral group ring of Φ/Φ' by $Z(\Phi/\Phi')$. The following result was proved in [2].

PROPOSITION 1. *Φ' as a natural $Z(\Phi/\Phi')$ module (in multiplicative notation) is defined by generators $[a_i, a_j]$ for all $i < j$ and relations*

$$[a_i, a_j]^{(1-a_k)}[a_i, a_k]^{-(1-a_j)}[a_j, a_k]^{(1-a_i)} = 1$$

for all $i < j < k$.

In the above and throughout this paper $[x,y]^{u+v}$ means $[x,y]^u[x,y]^v$ and $[x,y]^{-u}$ means $[y,x]^u$. The above relation, as usual, will be referred to as the Jacobi identity, and as is well known, is in fact a law in the metabelian variety.

Most important for our purposes is the result due to W. Magnus that any commutator of Φ can be written uniquely as a product of the following commutators (and their inverses) modulo any term Φ_m of the lower central series of Φ;

$$[a_{i_1}, \cdots, a_{i_m}], \quad i_1 > i_2 \leqq \cdots \leqq i_m, \tag{2.1}$$

for $2 \leqq m < n$. Thus for $n \geqq 2$, these commutators generate Φ'/Φ_n, and more particularly, the commutators of weight n occuring among the above set of commutators form a basis for the free abelian group Φ_n/Φ_{n+1}. For proofs one may consult H. Neumann [10]. (Note: If the above commutators were considered as commutators of the free group F with free generators $a_1, \cdots, a_q$, they would be the basic commutators—those commutators which arise in the collecting process of P. Hall [5]—which do not collapse when considered as elements of Φ. It should be observed that although for any fixed n these commutators generate Φ'/Φ_n and $\bigcap_1^\infty \Phi_n = 1$, these commutators do not generate Φ' (H. Neumann [10]).) The above commutators (2.1) of Φ will be referred to as the basic commutators.

As a final remark in this section we state the following well-known result which is used implicitly throughout this paper.

LEMMA 1. *$F'' \subset F_4$ but $F'' \not\subset F_5$. Hence $F/F_i \cong \Phi/\Phi_i$ for $i = 2,3,4$, but $F/F_i \not\cong \Phi/\Phi_i$ for $i > 4$.*

3. **Preliminary remarks concerning the automorphism groups.** The group of those automorphisms of a group G which induce the identity automorphism on a

quotient group K of G, will be denoted by $A(G;K)$. Thus the group of IA-automorphisms of G is denoted by $A(G;G/G_2)$. Suppose G is written as a quotient group $G=H/N$, where N is characteristic in H. The subgroup of $A(H/N;K)$ consisting of those automorphisms induced from $A(H;1)$ will be denoted by $A^*(H/N;K)$. We will write $A(G)$ for $A(G;1)$, and correspondingly $A^*(H/K)$ for $A^*(H/K;1)$. The following well-known lemmas are used implicitly throughout the paper. In the statements we may confine our attention to groups G which are either free groups or free metabelian groups of finite rank q, although the lemmas of course apply to a much larger class of groups.

LEMMA 2. *Assume $G=F$ or $G=\Phi$. Then $A^*(G/G_2)=A(G/G_2)$. Hence if N is a characteristic subgroup of G contained in G_2, and $A^*(G/N;G/G_2) = A(G/N;G/G_2)$, then $A^*(G/N)=A(G/N)$.*

LEMMA 3. *The kernel $K=A(G/G_n;G/G_{n-1})$ of the natural map from $A(G/G_n)$ into $A(G/G_{n-1})$ is free abelian, in fact isomorphic to the q-fold direct product of G_{n-1}/G_n. If $G=F$, free of rank q, then K has rank $q\Psi_q(n-1)$; if $G=\Phi$, free metabelian of rank q, then K has*

$$\text{rank } q(n-2)\binom{n+q-3}{q-2}.$$

The rank of K can easily be computed. If $G=F$, then rank $K=q\Psi_q(n-1)$, where Ψ is the Witt number (see §2). If $G=\Phi$, then rank K is q times the number of basic commutators of Φ of weight $n-1$. For positive integers s,t, let $N(s,t)$ be the number of (ordered) nonnegative integer solutions $m_1,\cdots,m_s$ of the equation $m_1+m_2+\cdots+m_s=t$. Then the number of basic commutators of Φ of weight $m-1$ is $\sum_{i=1}^{q-1} iN(i+1,m-3)$. $N(s,t)$ may be viewed as the number of ways of inserting $s-1$ separating zeros in a string of t ones, and hence is the binomial coefficient

$$\binom{t+s-1}{s-1}.$$

Thus we have

$$\sum_{i=1}^{h} iN(i+1,m) = \sum i\,\frac{(m+i)!}{m!\,i!} = \sum\frac{(m+i)!}{m!(i-1)!} = \sum\frac{(m+1)(m+i)!}{(m+1)!(i-1)!}$$
$$=(m+1)\sum_{j=0}^{h-1}\binom{m+1+j}{m+1}=(m+1)\binom{m+1+h}{m+2}$$
$$=(m+1)\binom{m+h+1}{h-1}.$$

Hence for $G=\Phi$ in Lemma 3, we have

$$\text{rank } K = q\sum_{1}^{q-1} iN(i+1,n-3)=q(n-2)\binom{n+q-3}{q-2}.$$

Let Φ be freely generated by $a_1, \cdots, a_q$. Then a faithful representation of Φ by 2×2 matrices due to W. Magnus [6] immediately yields a faithful representation of the IA-automorphism group of Φ as $q \times q$ matrices, over R, the group ring of Φ/Φ' over the integers. A description of the representation may be found in §3 of [2]. We indicate here the essential details.

If α is an IA-automorphism of Φ and

$$a_i\alpha = a_i \prod_{j<k} [a_j, a_k]^{C_{ijk}}, \tag{3.1}$$

where the C_{ijk} are polynomials in the commuting variables a_h^{+1} $(h = 1,2,\cdots,q)$, then

$$M(\alpha) = I + N(\alpha) \tag{3.2}$$

where I is the identity matrix and $N(\alpha) = (d_{uv})$ is given by

$$d_{uv} = s_u \left[\sum_{h>v} (1 - s_h)C_{uvh} - \sum_{h<v} (1 - s_h)C_{uhv} \right]. \tag{3.3}$$

(The polynomials C_{uvh} are the same as above except that the names of the variables are changed from $a_l^{\pm 1}$ to $s_l^{\pm 1}$.)

Remark. If the Fox derivatives [4] are evaluated in the abelianized Magnus ring, then the matrix $M(\alpha)$ associated with the automorphism α is just the Jacobian:

$$M(\alpha) = \left(\frac{\partial(a_i\alpha)}{\partial a_j} \right).$$

Some readers may prefer to view the representation in this manner. We will not use this and hence do not assume any knowledge of Fox derivatives.

We will henceforth refer to the ring of finite polynomials in the $s_i^{\pm 1}$ as S. Since det $M(\alpha)$ is a unit in S, we have det $M(\alpha) = \prod_{i=1}^q s_i^{j_i}$. Moreover, for an IA-automorphism α, with $a_i\alpha = a_i x_i$, x_i in F', it is known (see Fox [4]) that $\sum_j (\partial x_i/\partial a_j)(a_j - 1) = x_i - 1 \equiv 0$ modulo F'. This motivates the following proposition:

Proposition 2. *A necessary and sufficient condition for a $q \times q$ matrix $M(\alpha) = (a_{ij})$ over S to represent an IA-automorphism of the free metabelian group of rank q is that*

(i) $$\det M(\alpha) = \prod_{i=1}^{q} s_i^{j_i}, \quad \textit{where the } j_i \textit{ are integers}$$

and

(ii) $$\sum_{j=1}^{k} a_{ij}(1 - s_j) = 1 - s_i, \quad i = 1,2,\cdots,q.$$

The proof of Proposition 2 may be found in §3 of [2].

Let us denote as Σ the ideal in S generated by the $\sigma_i = 1 - s_i$ $(i = 1,2,\cdots,q)$. We need the following lemma.

LEMMA 4. *If* $\alpha \in A(\Phi; \Phi/\Phi_n)$, *then* $M(\alpha) \equiv I \pmod{\Sigma^{n-1}}$.

Proof. The basic commutators (2.1) of weight n (listed in §2) form a basis for Φ_n modulo Φ_{n+1}. Hence α maps the generators a_i of Φ into $a_i x_i$, where x_i can be written as a product of basic commutators of weight n (and their inverses) modulo lower terms of the lower central series. But this means that the polynomials C_{uvh} (defined in (3.1)) are in the ideal Σ^{n-2}. Thus by (3.2) and (3.3) Lemma 4 follows.

4. **Induced automorphisms of free groups.** Throughout this section, let F be a free group of rank q with free generators $a_1, \cdots, a_q$. We will show that all automorphisms of F/F_3 are induced by automorphisms of F. Our first lemma is somewhat more since it characterizes the kernel of the natural map of $A(F; F/F_2)$ into $A(F/F_3)$. These results, as indicated earlier, also appear in Andreadakis [1].

LEMMA 5. *Let K be the kernel of the natural homomorphism of $A(F; F/F_2)$ into $A(F/F_3)$. Then $K = A(F; F/F_2)'$, the commutator subgroup of $A(F; F/F_2)$.*

Proof. Let F have the representation in the free Magnus ring F^* and consider each element of $A(F; F/F_2)$ as a ring automorphism. Let us denote $x_i x_j - x_j x_i$ by (x_i, x_j) and denote by Σ the ideal generated by the x_i.

Let k be one of the Nielsen-Magnus generators k_{ij} or k_{ijl} of the IA-automorphism group of F as defined in the introduction. Passing to F^*/Σ^3, this gives a set of generators $\bar{k}$ for the image of $A(F; F/F_3)$ in $A(F/F_3; F/F_2)$, of the form

$$\bar{k}: \begin{cases} x_i \to x_i \pm (x_j, x_k) & \pmod{\Sigma^3}. \\ x_r \to x_r, & r \neq i. \end{cases}$$

From this, it is clear that $A^*(F/F_3; F/F_2)$ is free abelian on the $\bar{k}$, whence the kernel K contains the commutator subgroup of $A(F; F/F_2)$. Now using Magnus' Theorem, it follows that if α is in K, then the image of a generator x_j under α does not contain terms of dimension 2. From this remark it follows easily that K oincides with the commutator subgroup of $A(F; F/F_2)$. This proves Lemma 5.

THEOREM. *Under the natural mapping, we have*

$$A(F; F/F_2)/A(F; F/F_2)' \cong A(F/F_3; F/F_2).$$

Proof. Call the left-hand side B. Then, by Lemma 5, B is isomorphic to a subgroup of $A(F/F_3; F/F_2)$ and indeed B is a direct summand of $A(F/F_3; F/F_2)$. By the Nielsen-Magnus theorem, it is clear that B is free abelian of rank $\frac{1}{2}q^2(q-1)$. On the other hand, Lemma 3 shows that $A(F/F_3; F/F_2)$ has the same rank as B, and hence coincides with B.

As a corollary, we have

THEOREM 1. *All automorphisms of F/F_3 are induced by automorphisms of F.*

Proof. The above theorem states that all IA-automorphisms of F/F_3 are induced by automorphisms of F and hence we can apply Lemma 2 to conclude that all automorphisms of F/F_3 are induced by automorphisms of F.

COROLLARY. *Let V be a characteristic subgroup of F contained in F_3. Let $R = F/V$. Then any automorphism of R/R_3 is induced by an automorphism of R.*

In particular $R = \Phi$ if we take V to be F''. In the following section, we will see that not all automorphisms of Φ/Φ_4 ($\cong F/F_4$) are induced by automorphisms of Φ and hence certainly not of F. The information we get will be quite specific as to which automorphisms of Φ/Φ_4 are induced by automorphisms of Φ and in Section 6 we will show that these automorphisms are the same as those induced by automorphisms of F. This will therefore give us complete information as to which automorphisms of F/F_4 are induced by automorphisms of F. But nothing is known, apart from some obvious examples, about which automorphisms of F/F_n, for $n > 4$, are induced by automorphisms of F. (However cf. Andreadakis [1].) This is an open problem which appears to be very difficult. The corresponding problem of which automorphisms of Φ/Φ_n are induced by automorphisms of Φ will be solved in the next section.

5. **Induced automorphisms of free metabelian groups.** The group $A^* = A^*(\Phi/\Phi_n; \Phi/\Phi_{n-1})$ is the kernel of the natural map from $A^*(\Phi/\Phi_n)$ onto $A^*(\Phi/\Phi_{n-1})$, whence the question of which elements of $A(\Phi/\Phi_n)$ are induced by automorphisms of Φ, for $n = 2, 3, \cdots$, reduces to the same question for successive $A = A(\Phi/\Phi_n; \Phi/\Phi_{n-1})$. By Lemma 3, the free abelian group A has

$$\text{rank } q(n-2)\binom{n+q-3}{q-2}.$$

Here we undertake to compute the rank of its subgroup A^*. Throughout this section the letters A and A^*, without any accompanying parentheses, will mean the above groups.

If β is in $A(\Phi/\Phi_n; \Phi/\Phi_2)$, then (cf. Lemma 4), we may assume β is given by

$$a_i\beta = a_i \prod_{j<k} [a_j, a_k]^{A_{ijk}}, \tag{5.1}$$

where the A_{ijk} are (ordinary) integral polynomials of degree $n-3$ in the $(1-a_h)$ since monomials in the $(1-a_h)$ of degree higher than $n-3$ correspond to terms lower down in the lower central series. Also we might observe that the polynomials A_{ijk} are uniquely determined when the $a_i\beta$ are written in terms of basic commutators.

Suppose α in $A(\Phi)$ induces β in A, and has matrix $M(\alpha) = I + N(\alpha)$. By Lemma 4, we know that $N(\alpha)$ has all its entries in Σ^{n-2}. Similarly, $N(\alpha) \equiv 0 \pmod{\Sigma^{n-1}}$ if and only if α induces the identity mod Φ_n. For α in $A(\Phi; \Phi/\Phi_{n-1})$, let $N_1(\alpha)$ be the image of $N(\alpha)$ in $\Sigma^{n-2}/\Sigma^{n-1}$. Since $I + N(\alpha\alpha') = (I + N(\alpha))(I + N(\alpha'))$, reducing modulo Σ^{n-1} we have $N_1(\alpha\alpha') = N_1(\alpha) + N_1(\alpha')$. The kernel of the map from $A(\Phi; \Phi/\Phi_{n-1})$ into the additive group of matrices $N_1(\alpha)$ is exactly $A(\Phi; \Phi/\Phi_n)$. Thus N_1 is an isomorphism from A^* into an additive group η_1 of matrices over $\Sigma^{n-2}/\Sigma^{n-1}$. To compute the rank of A^* we compute that of η_1.

If β is in A, we define $N_1(\beta)$ in the obvious manner. That is, assuming β is given as in (5.1), we replace a_i by s_i in the A_{ijk} to define $N(\beta)$ as in (3.2) and (3.3) of §3 and thus determine the corresponding $N_1(\beta)$. Here the A_{ijk} play the role of the C_{ijk} in (3.2) and (3.3). Our immediate goal is to show that for β in A, β is in A^* if and only if trace $N_1(\beta) = 0$. This enables us to compute the rank of η_1 and immediately yields Theorem 3 (cf. the introduction) for which we will give a more explicit presentation later. This also easily yields the following equivalent formulation of Theorem 2 (cf. introduction):

THEOREM 2. *If $q \geqq 2$, $n \geqq 4$, then* rank A − rank A^* = *number of monomials of degree $n-2$ in the σ_i $(i = 1,2,\cdots,q)$.*

We begin as follows:

LEMMA 6. *Suppose α is in $A(\Phi)$. Then for $n \geqq 4$, α is in $A^*(\Phi/\Phi_n; \Phi/\Phi_{n-1})$ if and only if* det $M(\alpha) = 1$.

Proof. By Proposition 2, it is known that det $M(\alpha)$ is a monomial unit. Since $n \geqq 4$, applying Lemma 4, we have $M(\alpha) \equiv I \pmod{\Sigma^2}$ and hence det $M(\alpha) \equiv 1 \pmod{\Sigma^2}$. But det $M(\alpha) = \prod_1^q s_i^{j_i} = \prod_1^q (1-\sigma_i)^{j_i}$ and unless $j_1 = j_2 = \cdots = j_q = 0$, this expression contains linear terms in the σ_μ which contradicts the above. This proves the lemma.

Now we seek to determine the group $A^* = A^*(\Phi/\Phi_n; \Phi/\Phi_{n-1})$ for $n = 2,3,\cdots$. When $n = 2$, then $\Phi/\Phi_1 = 1$, $\Phi/\Phi_2 = F/F_2$, free abelian, and it was shown by Nielsen that the full unimodular group of automorphisms of F/F_2 is induced from automorphisms of F. When $n = 3$, it follows from Theorem 1 and its Corollary that every automorphism of Φ/Φ_3 is induced from an automorphism of Φ. Suppose henceforth $n \geqq 4$. These are precisely the cases for which Lemma 6 applies. Thus if α in $A(\Phi)$ is given in the manner of (3.1) and induces β in A^* given in the manner of (5.1), we may assume det $M(\alpha) = 1$. But, det $M(\alpha) = 1 +$ trace $N(\alpha)$ + terms quadratic or higher in the C_{ijk}. Modulo Σ^{n-1} this reduces to the condition that trace $N(\alpha) \equiv 0$, that is, that trace $N_1(\beta) = 0$. This condition that trace $N_1(\beta) = 0$ has the more explicit form

$$0 \equiv \sum_{i<k} \sigma_k A_{iik} - \sum_{i>k} \sigma_j A_{iji} \pmod{\Sigma^{n-1}}.$$

If we regard the A_{ijk} as polynomials in the σ_i, this becomes a system of linear conditions on the coefficients of the A_{ijk}. Write $A_{ijk} = \sum a_{ijk}(m)m$, summed over distinct monomials m. Then the condition trace $N_1(\beta) = 0$ reduces to the following system of linear conditions on the integer coefficients $a_{ijk}(m)$:

(5.2) $$0 = \sum_{i<k} a_{iik}(m/\sigma_k) - \sum_{i>j} a_{iji}(m/\sigma_j),$$

one such equation for each m of degree $n-2$ and clearly all these conditions are independent. We emphasize that both summations in (5.2) are over pairs of indices i, k (or i,j) such that $i<k$ (or $i>j$) and also over k (resp. j) such that σ_k divides m (or σ_j divides m).

Before proceeding further we make the following important observation: If P_{ijk} are any polynomials in the $a_i^{\pm 1}$, the mapping

(5.3) $$\begin{aligned} &a_i \to a_i \prod_{j<k} [a_j, a_k]^{P_{ijk}}, \quad j \neq i \neq k \\ &a_r \to a_r, \qquad r \neq i \end{aligned}$$

is an automorphism of Φ. (This is an easy consequence of Proposition 2, since in the translation to the matrix representation the determinant is 1.) Let us call a polynomial A_{ijk} where $i<j<k$, an exceptional polynomial. Thus, in the notation of (5.1), if all A_{ijk} are zero except for one exceptional polynomial (where this exceptional polynomial may be any prescribed integral polynomial in the σ_i), then β may be realized as an automorphism of Φ. (Remark: Notice that in the system (5.2), the coefficients of these exceptional polynomials do not appear. This is to be expected since in view of the above observation, the coefficients of these polynomials cannot be subjected to any constraints.)

Now taking the above observation into account and considering the linear system (5.2), it would be easy at this point to give an upper bound for the rank of A^* and hence conclude $A > A^*$. In fact it is clear already that not every α in A satisfies the above conditions (5.2), whence $A > A^*$. Specifically, the automorphism $a_1\alpha = a_1[a_1, a_2]^{\sigma_2^{n-3}}$, $a_i\alpha = a_i (i \neq 1)$, does not satisfy them and hence $A > A^*$ for $q \geqq 2$ and $n \geqq 4$. However, for the purpose of computing the rank of A^*, we must show that the necessary conditions (5.2) are also sufficient. This is contained in the following crucial lemma.

LEMMA 7. *For β in $A(\Phi/\Phi_n; \Phi/\Phi_{n-1})$, $n \geqq 4$, given in the manner of* (5.1), *one has β in A^* if and only if trace $N_1(\beta) = 0$.*

Proof. Given a solution of the linear system (5.2), we must show that there exists an automorphism of Φ having polynomials for which the terms of degree less than $n-3$ are zero and those of degree $n-3$ the given solution. It is enough to treat the case that only two of the coefficients actually appearing in a solution are different from zero (and hence two coefficients in a single equation of the system). This is because any solution of the system can be realized by adding such particular solutions and thus the automorphism corresponding to the general solution is realized by multiplying the automorphisms corresponding to the particular solutions.

In producing the desired automorphism α, we are at liberty to choose coefficients of degree larger than $n-3$ arbitrarily for all polynomials A_{ijk}, and, by virtue of the above observation, we may also choose coefficients of degree $n-3$ arbitrarily for all exceptional polynomials. We suppose at first that the two nonzero coefficients bear subscripts of the form i, i, k and j, j, h.

Case 1. $i=j$. We have $a_{iik}(m/\sigma_k) = -a_{iih}(m/\sigma_h)$, corresponding to

$$a_i\beta = a_i([a_i, a_k]^{\sigma_h}[a_i, a_h]^{-\sigma_k})^{m'},$$

with the $a_l \neq a_i$ fixed. (Here $m'\sigma_h\sigma_k = m$.) By the Jacobi identity, this is the same as $a_i\beta = a_i[a_h, a_k]^{+\sigma_i m'}$, hence realizable.

Case 2. $i \neq j$. We can suppose $i=1$, $j=2$, Let us first suppose $h=k$, in which case we may take $h=k=3$. Thus $a_{113}(m/\sigma_3) = -a_{223}(m/\sigma_3)$, corresponding to $a_1\beta = a_1[a_1, a_3]^{m'}$, $a_2\beta = a_2[a_2, a_3]^{m'}$, all other a_i fixed. (Here $m = \sigma_3 m'$.) We shall attempt to construct an automorphism α of Φ with all $C_{ijk}=0$ except C_{113}, C_{123}, C_{213}, C_{223}, and with $C_{113} \equiv -C_{223} \equiv m' \pmod{\Sigma^{n-2}}$. For the exceptional polynomials, we need only the condition that $C_{123} \equiv C_{213} \equiv 0 \pmod{\Sigma^{n-3}}$. (The notation is that of (3.1) in §3.) The condition that $\det M(\alpha) = 1$ takes the form

$$(*)\quad \begin{aligned} 1 = \det M(\alpha) &= 1 + \sigma_3(1-\sigma_1)C_{113} + \sigma_3(1-\sigma_2)C_{223} \\ &\quad + \sigma_3^2(1-\sigma_1)(1-\sigma_2)C_{113}C_{223} - \sigma_3^2(1-\sigma_1)(1-\sigma_2)C_{123}C_{213}. \end{aligned}$$

We define $C_{113} = m' - \sigma_2 m'$, $C_{123} = C_{113}$, $C_{213} = C_{223}$, and $C_{223} = -m' + \sigma_1 m'$. With these choices, one easily checks that (*) is satisfied, and we have $C_{113} \equiv -C_{223} \equiv m' \pmod{\Sigma^{n-2}}$ and $C_{123} \equiv C_{213} \equiv 0 \pmod{\Sigma^{n-3}}$. Thus α is an automorphism of Φ and induces β.

Case 3. $i=1$, $j=2$, $h \neq k$, $h=j$. We have $a_{112}(m/\sigma_2) = -a_{221}(m/\sigma_k)$, corresponding to $a_1\beta = a_1[a_1, a_2]^{\sigma_k m'}$, $a_2\beta = a_2[a_2, a_k]^{-\sigma_2 m'}$, all other a_j fixed. By the Jacobi identity, $[a_1, a_2]^{\sigma_k m'} = [a_1, a_k]^{\sigma_2 m'}[a_2, a_k]^{-\sigma_1 m'}$, and hence we are in Case 2 (since $k>2$ may be presumed to be 3).

Case 4. $i=1$, $j=2$, $h=3$, $k=4$. We have $a_{113}(m/\sigma_3) = -a_{224}(m/\sigma_4)$, corresponding to $a_1\beta = a_1[a_1, a_3]^{\sigma_4 m'}$, $a_2\beta = a_2[a_2, a_4]^{-\sigma_3 m'}$, all other a_i fixed. By the Jacobi identity, $[a_2, a_4]^{-\sigma_3 m'} = [a_2, a_3]^{-\sigma_4 m'}[a_3, a_4]^{-\sigma_2 m'}$, and hence we are once again in Case 2. This takes care of all cases with subscripts corresponding to *i,i,h* and *j,j,k*.

In view of the symmetry involved corresponding to $[a_j, a_k]^{-1} = [a_k, a_j]$ which is reflected in (5.2), we need not consider the case where the two nonzero coefficients bear the subscripts *i,j,i* and *h,k,h*. Hence to complete the proof of Lemma 7, we need only consider the case where the subscripts are of the form *i,i,h* and *j,k,j*. Here again, by an analysis similar to above and by symmetry considerations we need only consider the case $i=k=1$, $j=h=2$. We have $a_{112}(m/\sigma_2) = a_{212}(m/\sigma_1)$ corresponding to $a_1\beta = a_1[a_1, a_2]^{\sigma_1 m'}$, $a_2\beta = a_2[a_1, a_2]^{\sigma_2 m'}$, all other a_i fixed. (Here $m = \sigma_1\sigma_2 m'$.) We shall attempt to construct an automorphism α of Φ with all $C_{ijk} = 0$ except C_{112} and C_{212}, where $C_{112} \equiv \sigma_1 m' \pmod{\Sigma^{n-2}}$ and $C_{212} \equiv \sigma_2 m' \pmod{\Sigma^{n-2}}$. The condition that $\det M(\alpha) = 1$ takes the form

$$(**) \qquad 1 = \det M(\alpha) = 1 + \sigma_2(1-\sigma_1)C_{112} - \sigma_1(1-\sigma_2)C_{212}.$$

We define $C_{112} = \sigma_1 m' - \sigma_1\sigma_2 m'$, $C_{212} = \sigma_2 m' - \sigma_2\sigma_1 m'$. Condition (**) is satisfied and we have $C_{112} \equiv \sigma_1 m' \pmod{\Sigma^{n-2}}$, $C_{212} \equiv \sigma_2 m' \pmod{\Sigma^{n-2}}$. Thus α is an automorphism of Φ which induces β. This complete the proof of Lemma 7.

THEOREM 2. *For $q \geqq 2$, $n \geqq 4$, we have*

$$\text{Rank } A^* = \text{Rank } A - \binom{n+q-3}{q-1}.$$

Hence

$$\text{Rank } A^* = q(n-2)\binom{n+q-3}{q-2} - \binom{n+q-3}{q-1}.$$

Proof. The rank of the system (5.2) in the coefficients $a_{ijk}(m')$ is the number of monomials of degree $n-2$ in the σ_i $(i = 1,2,\cdots,q)$ which is

$$N(q,n-2) = \binom{n+q-3}{q-1}.$$

Moreover, since the coefficients mentioned enter only once in this system and with coefficient ± 1, A/A^* is torsion free. That is $A = A^* + B$, direct sum of free abelian groups, where B has rank $N(q,n-2)$. This proves Theorem 2.

If one wishes, with the aid of Lemma 7, to compute Rank A^* by counting automorphisms, one must make careful use of Proposition 1 (§2). It might be illuminating to see exactly where this is needed. The number u of linearly independent solutions of (5.2) is easily seen to be $q(q-1)N(q,n-3) - N(q,n-2)$. To this number we must add the number v of independent automorphisms induced by automorphisms of the form (5.3) and which do not occur as linear combinations of the above u automorphisms. Let us first consider for $i<j<k$ those automorphisms of the form $a_i\beta = a_i[a_j,a_k]^{A_{ijk}}$, all other a_l $(l \neq i)$ fixed, where A_{ijk} is a monomial of degree $n-3$ which has no factor $(1-a_h)$ for $h<j$. Let us call the number of these automorphisms v'. Then Proposition 1 guarantees that these v' automorphisms are linearly independent and moreover none of these occur as a linear combination of the above u automorphisms. However, if A_{ijk} has a factor $(1-a_h)$ where $h<j$, then this is clearly a linear combination of the $u+v'$ automorphisms. (For example $a_2\beta = a_1[a_2,a_3]^{(1-a_1)m'}$, all other a_i fixed, is the same automorphism as $a_1\beta = a_1[a_1,a_2]^{-(1-a_3)m'}[a_1,a_3]^{(1-a_2)m'}$, all other a_i fixed, which is accounted for among the u automorphisms; while

$$a_1\beta = a_1[a_3,a_4]^{(1-a_2)m'},$$

all other a_i fixed, is the same as $a_1\beta = a_1[a_2,a_3]^{-(1-a_4)m'}[a_2,a_4]^{(1-a_3)m'}$, all other a_i fixed, which is accounted for among the v' automorphisms.) Hence $v = v'$. A short computation shows that $v = q\{\sum_1^{q-2} iN(i+1, n-3)\}$. Thus

$$u + v = q\left\{\sum_1^{q-1} iN(i+1, n-3)\right\} - N(q, n-2) = \operatorname{rank} A - \binom{n+q-3}{q-1}.$$

COROLLARY 1. *If Φ has rank $q \geqq 2$, then not all automorphisms of Φ/Φ_n ($n \geqq 4$) are induced by automorphisms of Φ.*

Since $\Phi/\Phi_4 \cong F/F_4$, we have

COROLLARY 2. *If F has rank $q \geqq 2$, then not all automorphisms of F/F_4 are induced by automorphisms of F.*

COROLLARY 3. *If F has rank $q \geqq 2$, then not all automorphisms of F/F_n are induced by automorphisms of F, for $n \geqq 4$.*

Proof. This follows from Corollary 2 since any automorphism of F/F_n is induced by an automorphism of F/F_j for $j > n$. (See, for example Mostowski [7].)

We give a table of values for the ranks of A and A^* in the case that $q = 3$, with $n = 4, 5, \cdots, 8$.

rank of $\Phi = 3$

n	rank $A = 3(n-2)n$	rank $A^* = (5n^2 - 11n)/2$	Difference $= \binom{n}{2}$
4	24	18	6
5	45	35	10
6	72	57	15
7	105	84	21
8	144	116	28

We next use the method of proof of Lemma 7 to obtain an explicit set of generators for $A^*(\Phi/\Phi_n; \Phi/\Phi_2)$, $n \geqq 4$. Let $m \geqq 4$. Let $k_{ijk}(P)$ denote the automorphism of Φ defined by

$$k_{ijk}(P)\colon a_i \to a_i[a_j, a_k]^P,\ a_r \to a_r \quad \text{for } r \neq i,$$

where i, j and k are distinct and P is any monomial in the $(1 - a_h)$ $(h = 1, 2, \cdots, q)$ of degree $m - 2$.

Let $k_{ik}(P)$ denote the automorphism of Φ defined by

$$k_{ik}(P)\colon \quad \begin{aligned} a_i &\to a_i[a_i, a_j]^P[a_i, a_j, a_k]^{-P}[a_k, a_j]^P[a_k, a_j, a_k]^{-P}, \\ a_k &\to a_k[a_i, a_j]^{-P}[a_i, a_j, a_i]^P[a_k, a_j]^{-P}[a_k, a_j, a_i]^P, \\ a_r &\to a_r \quad \text{for } r \neq i, \end{aligned}$$

where i,j,k,m and P are as above. Finally let $k'_{ij}(Q)$ denote the automorphism of Φ defined by

$$k'_{ij}(Q): \begin{array}{lll} a_i & \to & a_i[a_i, a_j, a_i]^Q[a_i, a_j, a_i, a_j]^{-Q}, \\ a_j & \to & a_j[a_i, a_j, a_j]^Q[a_i, a_j, a_i, a_j]^{-Q}, \\ a_r & \to & a_r \quad \text{for } r \neq i, \end{array}$$

where i,j and k are as above, but Q is now any monomial in the $(1 - a_h)$ degree $m - 3$.

THEOREM. 3. *For $n \geqq 4$, $A^*(\Phi/\Phi_n; \Phi/\Phi_2)$ is generated by the automorphisms induced by the $k_{ijk}(P)$, $k_{ik}(P)$, $k'_{ij}(Q)$ as above, for all $m, 4 \leqq m \leqq n$.*

Proof. $k_{ijk}(P)$ defined above are the exceptional automorphisms mentioned earlier; if $i < k < j$, then $k_{ik}(P)$ defined above are the automorphisms constructed in Case 2 of Lemma 7; and the $k'_{ij}(Q)$ defined above are the final automorphisms constructed in Lemma 7. If we now disregard the ordering of the indices, then it is easy to see that any case not specifically considered in Lemma 7, will after perhaps an application of the Jacobi identity, fall into a product of the above automorphisms. With these remarks the proof of Theorem 2 is now clear.

It should be remarked that by taking any minimal generating set of solutions of (5.2) and by constructing the corresponding automorphisms of Φ, one arrives at a minimal set of automorphisms of Φ which generate $A^*(\Phi/\Phi_n; \Phi/\Phi_2)$. Furthermore, the specific automorphisms of Φ which are then constructed are by no means uniquely determined.

6. **Proof of Theorem 4.** Througout this section we assume that the canonically isomorphic groups Φ/Φ_4 and F/F_4 are identified. We will use the methods developed in the previous section to first show that

$$A^*(\Phi/\Phi_4; \Phi/\Phi_3) = A^*(F/F_4; F/F_3)$$

and then we will prove $A^*(\Phi/\Phi_4) = A^*(F/F_4)$.

LEMMA. 8. *If Φ has rank $q \geqq 2$, then $A^*(\Phi/\Phi_4; \Phi/\Phi_3) = A^*(F/F_4; F/F_3)$.*

Proof. The IA-automorphisms of Φ/Φ_4 which are induced by automorphisms of Φ are those which correspond to solutions of the system (5.2), plus the exceptional automorphisms mentioned in the previous section. Hence to prove Lemma 8, we must show that any automorphism of Φ/Φ_4 which corresponds to a solution of (5.2) is induced by an automorphism of F. We are in the case $n = 4$ of (5.2), where there is one condition for each monomial m of degree 2. If $q = 2$, the result is obvious since the automorphisms in question are inner. If $q \geqq 3$, as indicated in the proof of Lemma 7, we may reduce the situation to the case $q = 3$, by appropriate use of the Jacobi identity. Hence we may confine our attention to the case $q = 3$. In this case, the system (5.2) becomes the following:

$$(6.1)\qquad \begin{aligned} -a_{212}(\sigma_1) - a_{313}(\sigma_1) &= 0 \\ a_{112}(\sigma_2) - a_{323}(\sigma_2) &= 0 \\ a_{223}(\sigma_3) + a_{113}(\sigma_3) &= 0 \\ a_{112}(\sigma_1) - a_{212}(\sigma_2) - a_{313}(\sigma_2) - a_{323}(\sigma_1) &= 0 \\ a_{113}(\sigma_1) + a_{223}(\sigma_1) - a_{212}(\sigma_3) - a_{313}(\sigma_3) &= 0 \\ a_{112}(\sigma_3) + a_{113}(\sigma_2) + a_{223}(\sigma_2) - a_{323}(\sigma_3) &= 0, \end{aligned}$$

where reading from top to bottom, each equation corresponds respectively to the monomial $\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_1\sigma_3, \sigma_2\sigma_3$, and $\sigma_1\sigma_2$. A solution of (6.1) in which all the $a_{ijk}(\sigma_h)$ are zero, except for two with values $+1$ or -1, will be called a generating solution and the corresponding automorphism of Φ/Φ_4 a generating automorphism. For example, a generating solution (determined by the first equation of (6.1)) is obtained by taking $a_{212}(\sigma_1) = 1$, $a_{313}(\sigma_1) = -1$, and all other $a_{ijk}(\sigma_h) = 0$. The corresponding automorphism is given by

$$a_1\bar{\alpha} = a_1 \; a_2\bar{\alpha} = a_2[a_1, a_2]^{(1-a_1)}, \; a_3\bar{\alpha} = a_3[a_1, a_3]^{-(1-a_1)}.$$

Now, in the notation of the introduction, $\bar{\alpha}$ is induced by the automorphism $[k_{213}, k_{312}]$ of F. The computation is straightforward, and becomes easy working modulo F_4, since we have that commutators commute and an element of F_3 is in the center modulo F_4. Thus for example, in all cases encountered, $[u, vw] = [u,v]\,[u,w]$ and an element of F_3 is left fixed by an IA-automorphism. To illustrate, the above automorphism leaves a_1 fixed and maps a_2 successively as follows,

$$\begin{aligned} a_2 &\to a_2[a_3, a_1] \to a_2[a_3[a_1, a_2], a_1] \equiv a_2[a_3, a_1]\,[[a_1, a_2], a_1] \\ &\to a_2[a_1, a_3]\,[a_3, a_1]\,[[a_1, a_2], a_1] = a_2[a_1, a_2]^{1-a_1}, \end{aligned}$$

and a similar computation for a_3 yields the result.

There are twelve generating automorphisms in all, but, in view of obvious symmetries, it will suffice to consider, in addition to the automorphism above associated with the first equation of (6.1), three more associated with the fourth equation of (6.1). We list these by specifying for each the nonzero coefficient in the corresponding generating solution, indicating in each case an automorphism of F which can be seen by direct calculation to induce the given automorphism.

$$\begin{matrix} a_{113}(\sigma_1) = 1 \\ a_{212}(\sigma_3) = 1 \end{matrix} \leftrightarrow [k_{213}, k_{12}],$$

$$\begin{matrix} a_{113}(\sigma_1) = 1 \\ a_{313}(\sigma_3) = 1 \end{matrix} \leftrightarrow [k_{31}, k_{13}],$$

$$\begin{matrix} a_{223}(\sigma_1) = 1 \\ a_{3\ 3}(\sigma_3) = 1 \end{matrix} \leftrightarrow [k_{213}, k_{32}].$$

In addition to these twelve automorphisms corresponding to a minimal set of generating solutions of (6.1), there are six more required for a basis for $A^*(\Phi/\Phi_4;\Phi/\Phi_3)$. These may be chosen to be the following six automorphisms of Φ/Φ_4:

$$a_i \to a_i\,[a_j,a_l]^{(1-at)}, \qquad i\neq j\neq l\neq i,$$
$$a_r \to a_r, \qquad r\neq i,$$

where $i=1,2,3$ and $t=j,l$. But these six automorphisms, if considered as mappings of F into itself, are clearly automorphisms of F. (In fact, they are the automorphisms $[k_{ijl},k_{it}^{-1}]$.) This completes the proof of Lemma 8.

THEOREM 4. *If Φ has rank $k\geqq 2$, then $A^*(\Phi/\Phi_4)=A^*(F/F_4)$.*

Proof. Let A (resp. B) denote $A^*(\Phi/\Phi_4)$ (resp. $A^*(F/F_4)$) and let N (resp. M) denote $A^*(\Phi/\Phi_4;\Phi/\Phi_3)$ (resp. $A^*(F/F_4;F/F_3)$). Then N (resp. M) is normal in A (resp. B) and by Lemma 8, $N=M$. But A/N (resp. B/N) is $A^*(\Phi/\Phi_3)$ (resp. $A^*(F/F_3)$) and by Theorem 1 (§4), $A^*(F/F_3)=A(F/F_3)$ and hence $A/N=B/N$. From this we conclude $A=B$. This completes the proof of Theorem 4.

Theorem 4 is rather surprising because, speaking loosely, one would expect that Φ has many more automorphisms than F. But if so, they do not show up as induced automorphisms of F/F_4. More explicitly, we have shown that if α is any automorphism of Φ, there exists an automorphism β of F inducing an automorphism $\bar\beta$ of Φ such that, for all x in Φ, $x\alpha\equiv x\bar\beta \pmod{\Phi_4}$. The question is, first, whether an analogous results holds modulo Φ_n for large n, and second, whether the same holds modulo $\Phi_\omega=1$, that is, whether every automorphism of Φ is induced by one of F. For example, can one decide whether the following automorphism of Φ, $a_1\alpha=a_1^2[a_2,a_3]a_1^{-1}$, all other a_i fixed, is induced by an automorphism of F?

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